SMALL BOUND FOR BIRATIONAL AUTOMORPHISM GROUPS OF ALGEBRAIC VARIETIES (WITH AN APPENDIX BY YUJIRO KAWAMATA)

DE-QI ZHANG

ABSTRACT. We give an effective upper bound of $|\mathrm{Bir}(X)|$ for the birational automorphism group of an irregular n-fold (with $n = 3$) of general type in terms of the volume $V = V(X)$ under an "albanese smoothness and simplicity" condition. To be precise, $|\mathrm{Bir}(X)| \le d_3 V^{10}$. An optimum linear bound $|\mathrm{Bir}(X)| \le \frac{1}{3} \times 42^3 V$ is obtained for those 3-folds with non-maximal albanese dimension. For all $n \ge 3$, a bound $|\mathrm{Bir}(X)| \le d_n V^{10}$ is obtained when alb_X is generically finite, $\mathrm{alb}(X)$ is smooth and $\mathrm{Alb}(X)$ is simple.

1. **Introduction**

We work over the field $\mathbb{C}$ of complex numbers. Let X be a normal projective n-fold of general type. X is *minimal* if the canonical divisor K_X is nef and X has at worst terminal singularities (see Kawamata-Matsuda-Matsuki [31], Kollar-Mori [34]). It is known that $|\mathrm{Aut}(X)| \le 42 \deg(K_X)$ when $\dim X = 1$ (Hurwitz), and $|\mathrm{Aut}(X)| \le (42K_X)^2$ when X is a minimal surface of general type (Xiao [49], [50]). See also works of Andreotti, Howard-Sommese [27], Huckleberry - Sauer [28], Corti [13] and probably many others.

For Gorenstein minimal n-folds X of general type, Szabo [42] gave a polynomial upper bound of $|\mathrm{Bir}(X)|$, though its degree is quite huge. See also works of Catanese - Schneider [9], Xiao [51], Cai [8] and Kovacs [35].

In this paper, we try to get a more realistic upper bound (of order) for the whole birational automorphim group $\mathrm{Bir}(X)$. If fact, we consider irregular varieties X and, *without* assuming the minimality of X, obtain a linear (resp. degree 10) bound in the volume $V(X)$ when $n = 3$ (resp. for all $n \ge 3$) if the albanese map is not generically finite onto a variety of general type (resp. is generically finite onto a smooth variety contained in the simple Albanese variety); for details, see below.

Our approach here of considering the albanese morphism $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is sort of the generalization of the idea as in Xiao [48] and Catanese-Schneider [9] to find a G-equivariant pencil. The universal property of $\mathrm{Alb}(X)$ guarantees that any action of a group G on X

The author is supported by an Academic Research Fund of NUS
2000 Mathematics Subject Classification 14J50, 14E07.

induces a canonical action on $\mathrm{Alb}(X)$ such that alb_X is G-equivariant though G might not act faithfully on the latter. When alb_X is not generically finite onto a variety of general type, Ueno's result [43] says that $Y := \mathrm{alb}_X(X)/B_0$ is of general type, where B_0 is the identity connected component of the subtorus of $\mathrm{Alb}(X)$ stabilizing $\mathrm{alb}(X)$ with respect to its translation action. So one can apply the weak positivity of the relative dualizing sheaf due to Fujita [16], Kawamata [30] and Viehweg [46], then use the volume $V(X)$ to give an optimum upper bound of $V(Y)V(F)$ and finally reduce to the cases of the fibre and base (F being the general fibre of $X \to Y$).

The "best" case where alb_X is generically finite onto $W = \mathrm{alb}_X(X)$ of general type, turns out to be the hardest one. It is supposed to be the "best" situation because then $|6K_X|$ is birational according to Chen - Hacon [10] Corollary 5.3 . It may then improve the coefficient in Szabo's bound, but not the power of $V = V(X)$.

Therefore, we have to explore closely good properties of the abelian variety $\mathrm{Alb}(X)$ as a torus. Certainly, after quotient away the translations, the image $\overline{G}$ of $G = \mathrm{Aut}(X) \to \mathrm{Aut}_{\mathrm{variety}}(\mathrm{Alb}(X))$ can be thought of as a subgroup of $\mathrm{GL}_{2q}(\mathbb{Z})$ with $q = q(X) = \dim \mathrm{Alb}(X)$, via the rational representation on the first integral homology of $\mathrm{Alb}(X)$. But Feit's upper bound for the order of a finite subgroup in $\mathrm{GL}_{2q}(\mathbb{Z})$ with $2q > 10$, is a huge number $(2q)!\,2^{2q}$ (see Friedland [15]) and it is even attainable by the orthogonal group $O_{2q}(\mathbb{Z})$. Or one can apply Jordan's Lemma 2.7 below, together with Xiao's linear bound for abelian subgroups, but then the constant J_{2q} depends on q and hence on the volume of X, rather than on the dimension of X.

So we have to catch the missing information of X when passing to $\mathrm{Alb}(X)$. To do so, we first give a linear bound of Betti numbers of X in terms of the volume of X in Theorem 1.3 below, then bound the exponent $\exp(G)$ using the classical Lefschetz fixed point formula in Lemma 2.4 below, and finally bound $|G|$ itself. In the process, one needs a technical finiteness condition on the fixed locus $\mathrm{Alb}(X)^{\overline{G}}$ as in Theorem 1.2. We remark that this condition is automatically satisfied when $\mathrm{Alb}(X)$ is simple. Hopefully, one will be able to remove this restriction and eventually have the desired small bound for all irregular varieties.

One reason of our considering irregular varieties is that these ones are expected to have a bigger $\mathrm{Aut}(X)$ compared with the general ones.

The advantage of using the albanese map for irregular varieties is that one could bound $\mathrm{Aut}(X)$ inductively by reducing either to the cases of the fibre and base of a fibration, or to the case where alb_X is generically finite onto a variety of general type (which is covered by Theorem 1.2 to some extent), since alb_X always carries an action of G on X over to actions on the base and fibre of alb_X. This method is applicable in all dimensions. See Theorem 1.6 below for details.

We now state the main results of the paper. When alb_X is not generically finite onto a variety of general type, one obtains the optimum linear bound; see the Remark in the Appendix; see 1.7.

Theorem 1.1. *Let X be a smooth projective 3-fold of general type with irregularity $q(X) \ge 4$. Let $V = V(X)$ be the volume of X and $G = \mathrm{Bir}(X)$ $(\ge \mathrm{Aut}(X))$ the birational automorphism group of X.*

(1) *Suppose that X is not of maximal albanese dimension, i.e.,* dim $\mathrm{alb}_X(X) < \dim X$. *Then $|G| \le \frac{1}{3} \times 42^3\, V$.*

(2) *More generally, suppose only that* $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ *is not generically finite onto a 3-fold of general type in* $\mathrm{Alb}(X)$*, i.e., the Kodaira dimension $\kappa(\mathrm{alb}_X(X)) < \dim X$. Then $|G| \le \frac{1}{3} \times 42^3\, V$.*

When alb_X is generically finite onto a variety of general type, i.e., $\kappa(\mathrm{alb}_X(X)) = \dim X$, one has the following slightly bigger bound, but the degree of the bound is still a constant independent of $\dim X$.

Theorem 1.2. *Let X be a smooth projective n-fold $(n \ge 3)$ of general type. Let $V = V(X)$ be the volume of X and $G = \mathrm{Bir}(X)$. Assume the following conditions:*

(i) alb_X *is generically finite onto W $(= \mathrm{alb}(X))$ of general type,*

(ii) *W is smooth, and*

(iii) *either $A := \mathrm{Alb}(X)$ is a simple abelian variety; or G induces an action on A such that the fixed locus A^g for every* $\mathrm{id} \ne g \in G$ *is a non-empty finite set unless g is a translation of A.*

Then there is a constant d_n (indepedent of X) such that $|G| \le d_n V^{10}$.

Combining Theorems 1.1 and 1.2, one obtains a reasonably small upper bound of $|\mathrm{Bir}(X)|$ for some irregular 3-folds of general type.

One ingredient for the proofs of results above is the following linear bound of Betti numbers in terms of the volume.

This might give another simple proof of Xiao's linear bound of $\mathrm{ord}(g)$ in terms of $V(X)$ for every $g \in \mathrm{Aut}(X)$ with $\mathrm{ord}(g)$ prime under the assumption that K_X is ample; see Remark 4.4.

Theorem 1.3. *Let X be a smooth projective n-fold with ample K_X. Then there is a constant a_n, independent of X, such that the Betti numbers $B_i(X) \le a_n K_X^n$ for all i.*

Remark 1.4. (1) The constants d_n and a_n in the results above are computable from the proof. The main contributors towards them are some of the existing constants: x_n in Xiao's Theorem 2.11, J_n the Jordan constant in Lemma 2.7, r_n of Angehrn - Siu in Lemma 2.1 and h_n of Heier in Lemma 4.2.

(2) Note that the degree $(= 10)$ of the polynoimal in $V(X)$ in Theorem 1.2, is independent of the dimension n of X. We propose the following, which is somewhat more general than the one in Xiao [49] for smooth and minimal X and for $\mathrm{Aut}(X)$:

Conjecture 1.5. There is a constant δ_n such that the following holds for every smooth projective n-fold of general type, where $V = V(X)$ is the volume:

$$|\mathrm{Bir}(X)| \le \delta_n V.$$

When $n \le 2$, Conjecture 1.5 is confirmed by Hurwitz, Xiao [49] and Xiao [50], and one can take $\delta_n = (42)^n$. Theorem 1.1 confirms the conjecture for $n = 3$ with some additional assumption. The result below is a further evidence for arbitrary dimension.

Theorem 1.6. *Let X and Y be smooth projective varieties of general type and let $f : X \to Y$ be a surjective morphism with connected general fibre F. Suppose that $G = \mathrm{Bir}(X)$ acts regularly and faithfully on X and G acts on Y so that f is G-equivariant.*

If Conjecture 1.5 holds for both Y and F, then it also holds for X.

1.7. Terminology and Notation.

For a normal projective variety X of dimension n, we say that X is *minimal* if the canonical divisor K_X is nef and if X has at worst terminal singularities; see [31], [34].

For a divisor D on X, we define the *volume* of D as $V(D) = \limsup_{s\to\infty} h^0(X, sD)/(s^n/n!)$; see [36]. $V(X) := V(K_X)$ is called the *volume* of X, a birational invariant. Note that $V(X) = K_X^n$ when X is minimal.

Let X be a variety and $G \le \mathrm{Aut}(X)$. Let $G_x = \{g \in G \,|\, g(x) = x\}$ be the *stabilizer* subgroup. For $H \le G$, define the *fixed locus* $X^H := \{x \in X \,|\, h(x) = x \text{ for some id} \ne h \in H\}$. For $g \in G$, define the *fixed locus* $X^g := \{x \in X \,|\, g(x) = x\}$. Thus normally, $X^{\langle g\rangle} \supseteq X^g$.

For an abelian variety A, denote by $\mathrm{Aut}_{\mathrm{variety}}(A)$ the set of automorphisms of A as a variety, and $\mathrm{Aut}_{\mathrm{group}}(A)$ the subgroup of bijective homomorphisms.

When $H \le G$ and $g \in G$, denote by $C_H(g) = \{h \in H \,|\, gh = hg\}$ the *centralizer*. Define the *exponent* $\exp(G) = \mathrm{lcm}\{\mathrm{ord}(g) \,|\, g \in G\}$.

Acknowledgment. This work was started when I was visiting National Taiwan University in the summer of the year 2005. I am very grateful to Professor Alfred Chen for the discussion at the initial stage, Professor Kawamata for kindly writing an Appendix with which the old bound of degree 2 in the old version of Theorem 1.1 has been optimised to the current one, Professor Oguiso for the reference [44], and the referee for going the extra mile to help me in removing the ambiguous parts and better the presentation of the paper.

2. **Preliminary results**

We first show how K_X^n controls (or is controlled by) other invariants linearly. For its generalization to arbitrary n, see Proposition 4.1. We

remark that in the assertion (4) below, one can take $r = 2+n(n+1)/2$ or bigger according to Angehrn-Siu [2], or Kollar [33] Theorem 5.8.

Lemma 2.1. *Let X be a Gorenstein minimal projective n-fold ($n \ge 2$) of general type and $f : X' \to X$ a resolution.*

(1) *Suppose $n = 2$. Then $q(X) \le p_g(X) \le \frac{1}{2}K_X^2 + 2$, and $K_X^2 \le 9(1 + p_g(X))$.*

(2) *Suppose $n = 3$. Then $p_g(X) \le K_X^3 + 2$ and $q(X) \le 2 + 50K_X^3$. Also $\chi(\mathcal{O}(K_X)) = f^*K_X.c_2(X')/24 \ge K_X^3/72 > 0$ and*

$$P_m(X) = \frac{1}{12}m(m-1)(2m-1)K_X^3 + (2m-1)\chi(\mathcal{O}(K_X)) \ \ (m \ge 2).$$

(3) *Suppose $n = 3$ and $p_g(X) > 0$. Then $K_X^3 \le 144p_g(X)$.*

(4) *Suppose $n \ge 2$. Then for every $r \ge 4$ such that $|rK_X|$ is base point free and $|(r+1)K_X|$ is non-empty we have $P_r(X) \le n + r^nK_X^n/2$.*

Proof. (1) Now X is smooth for "terminal" means smooth in dimension 2. Note that $0 < \chi(\mathcal{O}_X) = 1 - q(X) + p_g(X)$ for surfaces of general type. Then the first two inequalities follow from the Noether inequality. The last one follows from the Miyaoka-Yau inequality in [38] Theorem 1.1 and [52]: $K_X^2 \le 3c_2(X)$ and the calculation: $1 + p_g(X) \ge \chi(\mathcal{O}_X) = (K_X^2 + c_2(X))/12 \ge K_X^2/9$.

(2) By Chen [11] Theorem 3, we have $p_g(X) \le 2 + K_X^3$. By Lee [37], $|4K_X|$ is base point free and we take its general memeber S which is smooth (noting that $\mathrm{Sing}(X)$ is finite because X is terminal). Consider the exact sequence

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0.$$

Note that $H^i(X, \mathcal{O}_X(-S)) = 0$ for $i = 1, 2$ by Kawamata-Viehweg vanishing. Taking cohomology of the exact sequence, we get $q(X) = q(S)$. Now by (1), $q(S) \le p_g(S) \le \frac{1}{2}K_S^2 + 2$. Substituting in $K_S = (K_X + S)|S = 5K_X|S$ and $K_S^2 = 100K_X^3$, we get the first part of (2). The second part of (2) follows from the Riemann-Roch formula in Reid [41] and Miyaoka-Yau inequality.

(3) By the proof of Hacon [21] page 6, under the assumption that $p_g(X) > 0$, one has $\chi(\omega_X) \le 2p_g(X)$. Now (3) follows from (2).

(4) Consider first the case $\dim X = 2$. The system $|rK_X|$ is base point free for all $r \ge 4$, by Bombieri [7]. By (1), $\chi(\mathcal{O}_X) \le 1 + p_g(X) \le 3 + K_X^2/2$. The pluri-genus formula says that $P_r(X) = r(r-1)K_X^2/2 + \chi(\mathcal{O}_X) \le 3 + \frac{1}{2}(1 + r(r-1))K_X^2$. One can verify that (4) is true.

Now assume that $n \ge 3$. Let X_{n-1} be a general member of $|rK_X|$. By Kollar-Mori [34] Lemma 5.17, X_{n-1} is again terminal with $K_{X_{n-1}} = (1+r)K_X|X_{n-1}$. Let X_i be the the intersection of $n-i$ general members of $|rK_X|$. Inductively, we see that each X_i is terminal with nef and big $K_{X_i} = (1 + (n-i)r)K_X|X_i$. Also $C = X_1$ is a connected smooth curve; note that C is linearly equivalent to the nef and big divisor $rK_X|X_2$ on the terminal, which is smooth in dimension 2, surface X_2;

one may also refer to Hartshorne [25] Ch III, Exercise 11.3, for the connectedness of each X_i. Inductively, we obtain $h^0(X, rK_X) \le 1 + h^0(X_{n-1}, rK_X|X_{n-1}) \le \cdots \le (n-1) + h^0(C, rK_X|C)$, by considering the exact sequence of cohomologies coming from

$$0 \to \mathcal{O}_{X_{i+1}} \to \mathcal{O}_{X_{i+1}}(rK_X|X_{i+1}) \to \mathcal{O}_{X_i}(rK_X|X_i) \to 0.$$

The divisor $D = rK_X|C$ on C is special in the sense of Griffiths-Harris [19] page 251, because $K_C - D = (1 + (n-2)r)K_X|C \ge (r+1)K_X|C \ge 0$. Applying Clifford's Theorem [ibid], one has $h^0(C, D) \le 1 + \deg(D)/2$. Substituting this and $\deg(D) = rK_X.(rK_X)^{n-1} = r^n K_X^n$ into the above, we obtain (4). $\square$

In the case of generically finite surjective map between varieties of general type, the volume upstairs can control the volume downstairs and sometimes even the degree of the map. See also Lemma 6.1.

Lemma 2.2. *Let X and Y be smooth projective n-folds of general type and $f : X \to Y$ a generically finite surjective morphism.*

(1) *We have $V(X) \ge \deg(f)\ V(Y)$.*
(2) *Suppose further that $|K_Y|$ defines a generically finite rational map Φ. Then $V(Y) \ge \deg\Phi\ \deg\Phi(Y) \ge 1$ and $\deg(f) \le V(X)$.*

Proof. Let $\varepsilon > 0$. By Fujita's approximation as in Lazarsfeld [36] Theorem 11.4.4, after smooth modifications of X and Y, we may assume that $K_Y \sim_{\mathbf{Q}} H + E$ with H an ample $\mathbf{Q}$-divisor and E an effective $\mathbf{Q}$-divisor, such that $H^n = V(H) > V(Y) - \varepsilon$. By the ramification divisor formula, we have $V(X) \ge V(f^*H) = (f^*H)^n = \deg(f)H^n > \deg(f)(V(Y) - \varepsilon)$. Now (1) follows by letting ε tend to zero.

The first inequality in (2) is by the proof of Hacon-McKernan [22] Lemma 2.2; the rest now follows. $\square$

The assertion (1) below means that the volume controls the order of a free-acting group, while (2) says that sometimes $|G|$ can be bounded by its exponent $\exp(G)$. We denote by $\mathrm{cartier}(Y)$ the Cartier index, i.e., the smallest positive integer such that cK_Y is a Cartier divisor.

Lemma 2.3. *Let X be a smooth minimal projective n-fold of general type and $f : X \to Y$ a finite surjective morphism.*

(1) *Suppose that f is unramified. Then $\deg(f) = K_X^n/K_Y^n \le K_X^n$. In particular, if a subgroup $H \le \mathrm{Aut}(X)$ acts freely on X, then $|H| = K_X^n/K_Y^n \le K_X^n$, where $Y = X/H$.*
(2) *Suppose that a subgroup $G \le \mathrm{Aut}(X)$ has isolated fixed locus X^G. Let $f : X \to Y := X/G$ be the quotient map and $c = \mathrm{cartier}(Y)$ the Cartier index. Then $c \mid \exp(G)$ and $|G| \le cK_X^n \le \exp(G)\ K_X^n$.*

Proof. We note that K_X is nef and big and that K_X^n is a positive integer.

(1) follows from the fact that $K_X = f^*K_Y$.

(2) Denote by $G_x = \{g \in G \,|\, g(x) = x\}$ the stabilizer subgroup at the point $x \in X$. We may regard G_x as a subgroup of $\mathrm{GL}_n(T_{X,x}) \cong \mathrm{GL}_n(\mathbf{C})$. For the image $\overline{x} \in Y$ of x, the local Cartier index $c(\overline{x})$ equals the index of the quotient group $G_x/G_x \cap \mathrm{SL}_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})/\mathrm{SL}_n(\mathbb{C}) \cong \mathbb{C}^*$, which is cyclic and generated by the image $\overline{g}$ of some $g \in G_x$. Thus $c(\overline{x})$ divides $\exp(G_x)$ and also $\exp(G)$. Hence $c = \mathrm{cartier}(Y)$, which is the lcm of $c(\overline{x})$, also divides $\exp(G)$. Since cK_Y is Cartier, it can be written as $H_1 - H_2$ with very ample divisors H_j not passing through the isolated set $\mathrm{Sing}(Y)$ (the image of X^G). Then $cK_Y^n = (H_1 - H_2).K_Y^{n-1} = (K_Y|H_1)^{n-1} - (K_Y|H_2)^{n-1}$ is an integer. Since $f : X \to Y$ is etale outside the finite set $\mathrm{Sing}(Y)$, one has $K_X = f^*K_Y$. Hence $|G| = \deg(f) = K_X^n/K_Y^n = cK_X^n/cK_Y^n \le cK_X^n$. □

An application of the result below is given in Remark 4.4. We remark that the finiteness assumption below on X^g can be removed for surfaces as shown in Ueno [44].

Lemma 2.4. *Suppose that X is a smooth projective n-fold and g is an automorphism of finite order. Assume that X^g is finite.*

(1) *The Euler number of the fixed locus X^g is given by $e(X^g) = \sum_{i=0}^{2n} (-1)^i \mathrm{Tr}\; g^*|H^i(X,\mathbb{Z})/(\mathrm{torsion})$.*

(2) *One has $|e(X^g)| \le \sum_{i=0}^{2n} B_i(X)$, where $B_i(X)$ is the Betti number.*

Proof. (1) is the classical Lefschetz fixed point formula. For (2), let $\mathrm{ord}(g) = m$ and diagonalize $g^*|H^i(X,\mathbb{C}) = \mathrm{diag}[\zeta_m^{s_1}, \dots, \zeta_m^{s_{b_i}}]$ where $b_i = B_i(X)$ and $\zeta_m^{s_j}$ is an m-th root of 1. Thus $|\mathrm{Tr} g^*|H^i(X,\mathbb{Z})/\mathrm{torsion}| = |\sum_{j=1}^{b_i} \zeta_m^{s_j}| \le b_i$. This proves the lemma. □

The following is the relation between two stabilizers G_x and $(G/H)_{\overline{x}}$.

Lemma 2.5. *Suppose that a normal subgroup $H \trianglelefteq G$ acts freely on a variety X. Set $\overline{G} = G/H$ and $\overline{X} = X/H$.*

(1) *For every $x \in X$ and its image $\overline{x} \in \overline{X}$, the homomorphism between stabilizers $\varphi : G_x \to G_{\overline{x}}$ which is given by $g \mapsto \overline{g} = gH$, is an isomorphism.*

(2) *In particular, the fixed locus X^G is empty (resp. finite) if and only if $\overline{X}^{\overline{G}}$ is empty (resp. finite).*

Proof. To show that φ is surjective, suppose that $\overline{g}(\overline{x}) = \overline{x}$. Then $g(x) = h(x)$ for some $h \in H$. Thus $h^{-1}g \in G_x$ and $\overline{g} = \varphi(h^{-1}g)$. To show that $\mathrm{Ker}(\varphi)$ is trivial, suppose that $\varphi(g) = \mathrm{id} \in \overline{G}_{\overline{x}} \le \overline{G}$. So $g \in H$. Hence $g \in H \cap G_x = \{\mathrm{id}\}$ by the freeness of the H action on X. So $\mathrm{Ker}(\varphi)$ is trivial and the lemma is proved. □

Next are about nilpotent groups and relation between centralizers.

Lemma 2.6. *Let G be a finite group.*

(1) *If G is nilpotent, then $\exp(G)$ equals $\mu(G) := \max\{\mathrm{ord}(g) \,|\, g \in G\}$.*

(2) *Consider quotient groups inclusion* $K/H \trianglelefteq G/H$. *Let* $\tau \in K$ *and* $\overline{\tau} = \tau H \in G/H$. *Then* $|C_{K/H}(\overline{\tau})| \le |C_K(\tau)| \le |C_G(\tau)|$.

Proof. (1) is true for p-groups while G, being nilpotent, is a direct product of its Sylow subgroups (see Gorenstein [18] Theorem 3.5). So (1) follows.

(2) Write $C_{K/H}(\overline{\tau}) = L/H$. Consider the map: $\varphi : L \to \tau H$ ($g \mapsto g^{-1}\tau g$). Then $|L/C_L(\tau)| = |\mathrm{Im}(\varphi)| \le |H|$, so $|L/H| \le |C_L(\tau)|$. □

When $\mathrm{Aut}(X)$ fixes a point x, Jordan's lemma and its consequences below, together with Xiao's Theorem 2.11, will produce a linear bound $|\mathrm{Aut}(X)| \le J_n x_n K_X^n$.

Lemma 2.7. *The following are true.*

(1) *(***Jordan's lemma***) Let* $J_n = (n+2)!$ *(resp.* $J_n = n^4(n+2)!$*) for* $n > 63$ *(resp. for* $n \le 63$*). Then every finite group in* $\mathrm{GL}_n(\mathbb{C})$ *contains an abelian normal subgroup of index* $\le J_n$.

(2) *Suppose that* X *is an* n*-fold and* $G \le \mathrm{Aut}(X)$ *is a finite subgroup fixing a smooth point* $x \in X$. *Then* G *contains an abelian normal subgroup of index* $\le J_n$.

(3) *Suppose that* X *is a projective* n*-fold of general type and* $x \in X$ *a smooth point. If* $G \le \mathrm{Aut}(X)$ *fixes* x*, then* G *contains an abelian normal subgroup of index* $\le J_n$.

Proof. For (1), see Weisfeiler [47] page 5279 (for better bound) and Aljadeff- Sonn [3] page 353. The original number produced by Jordan was $J(n) = (49n)^{n^2}$. For (2), G can be regarded as a subgroup of $\mathrm{GL}_n(T_{X,x}) \cong \mathrm{GL}_n(\mathbb{C})$, so apply (1). For (3), $\mathrm{Aut}(X)$ and hence G are finite since X is of general type. Then apply (2). □

The result below has been proved by many authors. We only mention that the nefness of Ω_X^1 comes from the exact sequence below and the fact that quotient bundles of a nef bundle are again nef (see Hartshorne [25] Ch II, Theorem 8.17; Lazarsfeld [36] Proposition 6.1.2):

$$0 \to N_{X/A}^{\vee} \to \Omega_A^1 | X \to \Omega_X^1 \to 0.$$

Theorem 2.8. *Let* A *be an abelian variety and* $X \subset A$ *a subvariety of general type.*

(1) $|K_{\tilde{X}}|$ *defines a generically finite map, where* $\tilde{X} \to X$ *is any desingularization.*

(2) *Suppose that* X *is smooth. Then* Ω_X^1 *is nef and* K_X *is ample. Moreover,* $|K_X|$ *is base point free and* $\Phi_{|K_X|}$ *is a finite morphism.*

Proof. For the proof, see Ran [40] Corollary 2, or Abramovich [1]. See also Hartshorne [24], Ueno [43] and Griffiths-Harris [20] for earlier development. □

Here are some properties of simple abelian varieties to be used in §6.

Lemma 2.9. *Let A be a simple abelian variety.*

(1) *Every proper subvariety of A is of general type.*
(2) *If $\mathrm{id} \ne g \in \mathrm{Aut}_{\mathrm{variety}}(A)$ is not a translation, then A^g is a non-empty finite set.*
(3) *If $X \to A$ is a non-contant morphism from a projective variety X, then $q(\tilde{X}) \ge \dim(A)$, where $\tilde{X} \to X$ is any desingularization.*

Proof. (1) is just Ueno [43] Corollary 10.10.

(2) Suppose that g is not a translation. By Birkenhake-Lange [6] Lemma 13.1.1, we may assume that $\mathrm{id} \ne g \in \mathrm{Aut}_{\mathrm{group}}(A)$. Now $\mathrm{Ker}(g-\mathrm{id})$ is a proper subgroup of A, whence it is non-empty and 0-dimensional by the simplicity of A. So (2) follows (see [6] Formula 13.1.2).

(3) By the universal property of $\mathrm{Alb}(\tilde{X})$, the non-constant morphism $\tilde{X} \to A$ factors as $\tilde{X} \to \mathrm{Alb}(\tilde{X}) \to A$, where the latter map is a homomorphism (modulo a translation) and has image the translation of a non-trivial subtorus of A. This image is A by the simplicity of A. This implies (3) : $\dim(A) \le \dim \mathrm{Alb}(\tilde{X}) = q(\tilde{X})$. □

By verifying the case $\ell = 8$ (resp. $\ell = 12$) and using induction, we have (1) (resp. (2)) below to be used in Section 5.

Lemma 2.10. *Let A_k be the alternating group of order $k!/2$. Let $\ell = [k/4] = \max\{z \in \mathbb{Z} \,|\, z \le k/4\}$ so that $4\ell \le k \le 4\ell + 3$.*

(1) *If $\ell \ge 8$, then $|A_k| \le (4\ell+3)!/2 \le |A_{3\ell}|^{1.7}$.*
(2) *If $\ell \ge 12$, then $|A_k| \le |A_\ell|^8$.*

We end §2 with Xiao's linear bound for abelian subgroups:

Theorem 2.11. *(Xiao [51] Theorem 1) Let X be a smooth projective n-fold with nef and big K_X. Let $G \le \mathrm{Aut}(X)$ be an abelian subgroup. Then there exists a constant x_n, independent of X, such that $|G| \le x_n K_X^n$.*

3. G-equivariant fibrations; the proof of Theorem 1.6

In this section we consider G-equivariant fibration $f : X \to Y$. We will prove simultaneously Theorem 3.1 below and Theorem 1.6 in the Introduction.

Theorem 3.1. *Let X and Y be smooth projective varieties of general type and dimensions n and k, respectively. Suppose that a group G acts faithfully on X and acts anyhow on Y. Let $f : X \to Y$ be a G-equivariant surjective morphism with connected general fibre F. Suppose further that $|\mathrm{Bir}(Y)| \le a_1\, V(Y)^d$ and $|\mathrm{Bir}(F)| \le a_2\, V(F)^d$ for some positive integers a_i and d.*

Then we have:

$$|G| \le a\, V(X)^d, \qquad \text{with} \quad a = \frac{a_1 a_2}{(\binom{n}{k})^d} \ .$$

We now prove first Theorem 3.1 and late Theorem 1.6 as an application. Note that the general fibre F is also of general type since the Kodaira dimension $\kappa(X) = \dim X$ and by Iitaka's easy addition: $\kappa(X) \le \kappa(F) + \dim Y$. Since f is G-equivariant, we have the natural exact sequence:

$$1 \to K \to G \to \mathrm{Aut}(Y).$$

The group K acts trivially on Y and hence faithfully on the fibre F, so K can be regarded as a subgroup of $\mathrm{Aut}(F)$, where F is smooth and of general type. Now Theorem 3.1 follows from the Theorem in the Appendix:

$$|G| \le |K|\,|\mathrm{Aut}(Y)| \le |\mathrm{Aut}(Y)|\ |\mathrm{Aut}(F)| \le$$

$$|\mathrm{Bir}(Y)|\ |\mathrm{Bir}(F)| \le a_1 a_2\ V(Y)^d\ V(F)^d \le aV(X)^d.$$

Finally, Theorem 1.6 is a consequence of Theorem 3.1 and the inductive assumption, where we set $a_1 = \delta_{\dim Y}$, $a_2 = \delta_{\dim F}$ and $d = 1$.

Remark 3.2. Weaker versions of Theorems 1.6 and 3.1 can also be proved using Lemma 2.1, Kollar [32] Theorem 3.5 (ii) and Catanese-Schneider [9] pages 10-11, which are strengthened to the current form, thanks to the Appendix - an application of the weak positivity due to Fujita [16], Kawamata [30] and Viehweg [46].

4. **Bound of Betti numbers; the proof of Theorem 1.3**

In this section, for a smooth projective variety X, we shall bound the invariants like Betti numbers $B_i(X)$ and the Euler number $e(X)$, in terms of the volume of X. We first bound the irregularity and geometric genus.

Proposition 4.1. *Let X be a smooth projective n-fold ($n \ge 2$) with ample K_X. Then $p_g(X) \le e_n K_X^n$ and $q(X) \le e_n K_X^n$, where $e_n = \max\{n + r^n,\, 2 + \frac{1}{2}r^{n-2}(1 + (n-2)r)^2\}$ with $r = 2 + n(n+1)/2$.*

Proof. For $r = 2 + n(n+1)/2$, as in Lemma 2.1, $|rK_X|$ is base point free. Since K_X is ample, $\Phi = \Phi_{|rK_X|} : X \to \mathbb{P}^N$ with $N = P_r(X) - 1$, is a finite morphism. Hence $(rK_X)^n = \deg(\Phi)\deg\Phi(X) \ge \deg\Phi(X) \ge N - n + 1$ (see Griffiths-Harris [19] page 173). Thus $p_g(X) \le P_r(X) = N + 1 \le n + r^n K_X^n \le (n + r^n)K_X^n$.

As in Lemma 2.1, take the ladder $X = X_n \supset X_{n-1} \supset \cdots \supset X_2$, where X_i is the intersection of $n - i$ general members of $|rK_X|$, and a smooth i-fold with ample $K_{X_i} = (1 + (n-i)r)K_X|X_i$. Each X_{i-1} is a smooth ample divisor on X_i. By Lefschetz hyperplane section theorem, we have $q(X) = q(X_{n-1}) = \cdots = q(X_2)$. By Lemma 2.1, we conclude the proposition: $q(X_2) \le \frac{1}{2}K_{X_2}^2 + 2 = 2 + \frac{1}{2}r^{n-2}(1 + (n-2)r)^2 K_X^n$. $\square$

To bound the Betti numbers, we need the bound of Euler number $e(X)$ below.

Lemma 4.2. *Let X be a smooth projective n-fold.*

(1) *If the cotangent bundle Ω_X^1 is nef, then $0 \le (-1)^n e(X) = c_n(\Omega_X^1) \le K_X^n$.*

(2) *If K_X is ample then $|e(X)| \le h_n K_X^n$, where h_n is the constant, independent of X, in Heier* [26] *Proposition 1.7.*

Proof. (2) is proved in Heier [26] Proposition 1.7. For (1), by the nefness of Ω_X^1 and Demailly-Peternell-Schneider [14] page 31, we have $0 \le c_n(\Omega_X^1) \le c_1(\Omega_X^1)^n = K_X^n$. Note also that $c_n(\Omega_X^1) = (-1)^n c_n(T_X) = (-1)^n c_n(X) = (-1)^n e(X)$, where T_X is the tangent bundle; see for instance, Fulton [17] Example 3.2.13. This proves the lemma. □

4.3. Proof of Theorem 1.3

We will prove by induction on n. The case $n = 1$ is clear. When $n = 2$, it follows from Lemma 2.1 or 4.1. Indeed, note that $B_1(X) = 2q(X)$, $\chi(\mathcal{O}_X) \le 1 + p_g(X)$, $c_2(X) \le K_X^2 + c_2(X) = 12\chi(\mathcal{O}_X)$ and $B_2(X) = c_2(X) - 2 + 4q(X)$.

Suppose $n \ge 3$ and the theorem is true for all such k-folds with $2 \le k < n$. Let X_{n-1} be a general member of $|rK_X|$ with $r = 2 + n(n+1)/2$ (see Lemma 2.1). Note that $K_{X_{n-1}} = (1+r)K_X|X_{n-1}$ is ample and $K_{X_{n-1}}^{n-1} = r(1+r)^{n-1}K_X^n$. By induction, $B_i(X_{n-1}) \le a_{n-1}K_{X_{n-1}}^{n-1} = a'_n K_X^n$ where $a'_n = a_{n-1}r(1+r)^{n-1}$. By Lefschetz hyperplane section theorem, $B_i(X) = B_i(X_{n-1})$ for all $i \le n-2$ and $B_{n-1}(X) \le B_{n-1}(X_{n-1})$ (see Lazarsfeld [36] Theorem 3.1.17). Note also that $B_j(X) = B_{2n-j}(X)$ by Lefschetz duality. Thus $B_j(X) \le B_j(X_{n-1}) \le a'_n K_X^n$ for all $j \ne n$.

On the other hand, by Lemma 4.2, $h_n K_X^n \ge |\sum_{i=0}^{2n}(-1)^i B_i(X)|$. So $|B_n(X)| \le h_n K_X^n + |\sum_{i \ne n}(-1)^i B_i(X)| \le a_n K_X^n$ where $a_n = h_n + 2na'_n$. We are done by the induction. This proves Theorem 1.3.

Remark 4.4. For the X in Theorem 1.3, it might be possible to give a second proof of Xiao's linear bound (in terms of K_X^n) of $\mathrm{ord}(g)$ for every $g \in \mathrm{Aut}(X)$ with $\mathrm{ord}(g)$ prime and X^g finite. Indeed, note that the quotient map $X' = X - X^g \to X'/\langle g \rangle$ is unramified, and hence $\mathrm{ord}(g)$ divides the Euler number $e = e(X) - e(X^g)$, so e is bounded by $|e(X)| + |e(X^g)|$ (provided that $e \ne 0$) while the latter is bounded linearly by K_X^n (see Lemmas 2.4 and 4.2 and Theorem 1.3).

5. Automorphism groups of subvarieties of abelian varieties

In this section, we shall prove the following two results:

Theorem 5.1. *Let X be a smooth n-fold ($n \ge 3$) of general type contained in an abelian variety A of dimension q. Let G be a subgroup of $\{g \in \mathrm{Aut}_{\mathrm{variety}}(A) \,|\, g(X) = X\}$ such that the fixed locus A^g for every $\mathrm{id} \ne g \in G$ is a non-empty finite set unless g is a translation of A. Set $V = K_X^n$ (see Theorem 2.8).*

Then there is a constant d_n*, independent of* X *and* A*, such that* $|G| \le d_n q^{10-b} V^b \le d_n S^{10}$*, where* $S = \max\{V, q\}$ *and* $5 \le b \le 10$*.*

Corollary 5.2. *Let* A *be a simple abelian variety and* $X \subset A$ *a smooth* n*-dimensional* $(n \ge 3)$ *proper subvariety. Let* $G = \{g \in \mathrm{Aut}_{\mathrm{variety}}(A) \mid g(X) = X\}$ *be the stabilizer. Set* $V = K_X^n$ *(see Theorem 2.8). Then there is a constant* d'_n *(independent of* X *and* A*) such that* $|G| \le d'_n V^{10}$*.*

We fix an abelian variety A of dimension q and an n-dimensional smooth subvariety $X \subset A$ of general type. Let G be a subgroup of the group $\{g \in \mathrm{Aut}_{\mathrm{variety}}(A) \mid g(X) = X\}$. Note that $\mathrm{Aut}_{\mathrm{variety}}(A) = T \rtimes A_0$ (split extension) where $T = \{T_t \mid t \in A\}$ is the normal subgroup of translations and $A_0 = \mathrm{Aut}_{\mathrm{group}}(A)$ is the subgroup of bijective group-homomorphisms.

5.3. Assumption.

(i) $X \subset A$ is a smooth projective n-fold of general type in the abelian variety A of dimension q,

(ii) the subgroup $G \subseteq \{g \in \mathrm{Aut}_{\mathrm{variety}}(A) \mid g(X) = X\}$ contains no translations of A (we note that $G \le \mathrm{Aut}(X)$ is finite because X is of general type), and

(iii) the fixed locus A^g for every $\mathrm{id} \ne g \in G$ is a non-empty finite set.

We collect some information on the structure of G.

Lemma 5.4. *Suppose that* $G \subseteq \mathrm{Aut}(X)$ *and* $X \subset A$ *satisfy conditions in 5.3. Let* $\mathrm{id} \ne g \in G$*. Then we have:*

(1) *Let* G_0 *be the image of the composition* $G \subseteq \mathrm{Aut}_{\mathrm{variety}}(A) \to A_0$*. Then* $G \to G_0$ *is an isomorphism.*

(2) *For* $g \in G$ *and* g_0 *its image in* G_0*, we have* $|A^g| = |A^{g_0}| \ge |\{0\}| = 1$*.*

(3) *If* $H \le G$ *is abelian, then* H *is cyclic.*

(4) *For* $H \le G$*, if* $|H| = p^2$ *for some prime* p*, then* H *is cyclic.*

(5) *If* $|X^g| \ge 2$ *for some* $g \in G$*, then* $\mathrm{ord}(g) = p^s$ *for some prime* p*.*

(6) *If* g *has order* m *in* G *and if* ζ_m *denotes a primitive* m*-th root of* 1*, then there is a diagonalization* $g^* | H^0(A, \Omega_A^1) = \mathrm{diag}[\zeta_m^{s_1}, \dots, \zeta_m^{s_q}]$ *where each* $\zeta_m^{s_j}$ *is a primitive* m*-th root of* 1*.*

(7) *If* $g \in G$*, then the Euler function* $\varphi(\mathrm{ord}(g))$ *divides* $2q$*.*

(8) *If* p^2 *divides* $|G|$ *for some prime* p*, then there is an element* $g \in G$ *such that* $\mathrm{ord}(g) = p^2$*.*

(9) *If* $g \in G$*, then* $\mathrm{ord}(g)$ *divides* $4q^2\alpha(g)$*, where*

$$\alpha(g) = \prod_{\mathrm{prime}\, p \,|\, \mathrm{ord}(g),\, p^2 \nmid |G|} p.$$

(10) *If* $H \le G$*, then the exponent* $\exp(H)$ *divides* $4q^2\alpha(H)$*, where*

$$\alpha(H) = \prod_{\mathrm{prime}\, p \,|\, |H|,\, p^2 \nmid |G|} p.$$

Proof. 5.3 (ii) implies (1). Here (2), (3), (6) and (7) are consequence of 5.3 (iii) and Birkenhake - Lange [6] Lemma 13.1.1 and Propositions 13.2.2, 13.2.1 and 13.2.5. Our (4) is because a group of order p^2 is abelian and by (3). Now (5) follows from (1) and Birkenhake - Lange [6] Corollary 13.2.4. Also (8) is a consequence of (4) together with Sylow's theorem and a basic result on p-groups. For (9), write $\mathrm{ord}(g) = p_1^{t_1} p_2^{t_2} \dots p_u^{t_u}$ with $t_i \ge 1$. Then $\varphi(\mathrm{ord}(g)) = p_1^{t_1-1}(p_1 - 1) \dots p_u^{t_u-1}(p_u - 1)$. If $t_i \ge 2$, then $2(t_i - 1) \ge t_i$, and $p_i^{t_i}$ divides $(\varphi(\mathrm{ord}(g)))^2$ as well as $(2q)^2$ by (7). If $p^2 \mid |G|$, then $p \mid 2q$ by (8) and (7). Thus $\mathrm{ord}(g) \mid 4q^2\alpha(g)$ and (9) is proved. Our (9) implies (10). This proves the lemma. □

The centralizer lemma below is a key to the proof of Theorem 5.1.

Lemma 5.5. (**centralizer lemma**) *Suppose that $G \subseteq \mathrm{Aut}(X)$ and $X \subset A$ satisfy conditions in 5.3. Let $\tau \in G$ be of prime order p such that the fixed locus X^τ is non-empty. Let $V = K_X^n$ (see Theorem 2.8).*

(1) *Every prime factor $p_1 \neq p$ of the order $|C_G(\tau)|$ of the centralizer, divides $|X^\tau|(|X^\tau| - 1)$.*
(2) $\exp(C_G(\tau))$ *divides $4p^\varepsilon q^2 |X^\tau|(|X^\tau| - 1)$, where $\varepsilon = 0$ (resp. 1) when p^2 divides $|G|$ (resp. otherwise).*
(3) *One has $|C_G(\tau)| \le k_n q^2 V^{3+\varepsilon}$, where $k_n = \max\{J_n x_n,\, 4a_n^2 x_n^\varepsilon\}$; see Lemma 2.7 and Theorems 1.3 and 2.11.*

Proof. (1) Suppose $g \in C_G(\tau)$ has order equal to a prime $p_1 \neq p$. Write $|X^\tau| \equiv r$ with $0 \le r < p_1$. If $r = 0, 1$, then we are done. Suppose that $r \ge 2$. Then $|X^{\tau g}| \ge r \ge 2$. By Lemma 5.4, $pp_1 = \mathrm{ord}(\tau g)$ equals a prime power, a contradiction.

(2) follows from (1) and Lemma 5.4 (10).

(3) If $|X^\tau| = 1$, then $C_G(\tau)$ fixes the unique point x in X^τ. Hence $C_G(\tau) \le \mathrm{GL}_n(T_{X,x})$ and we apply Lemma 2.7 and Theorem 2.11 and obtain $|C_G(\tau)| \le J_n x_n K_X^n$. Suppose now that $|X^\tau| \ge 2$. By (2), Lemmas 2.3 and 2.4 and Theorem 1.3, we have $|C_G(\tau)| \le \exp(C_G(\tau)V \le 4p^\varepsilon q^2 |X^\tau|(|X^\tau| - 1)V \le 4p^\varepsilon q^2 (\sum_{i=0}^{2n} B_i(X))^2 V \le 4a_n^2 p^\varepsilon q^2 V^3$. By Theorem 2.11, $p \le x_n V$ and hence (3) follows. This proves the lemma. □

Theorem 5.1 should follow from the crucial proposition below.

Proposition 5.6. *Suppose that $X \subset A$ and $G \subseteq \mathrm{Aut}(X)$ satisfy the conditions in 5.3. Let $q = \dim(A)$ and $V = K_X^n$ (see Theorem 2.8). Then the there is a constant d_n, independent of X and A, such that $|G| \le d_n q^{10-b} V^b \le d_n S^{10}$, where $S = \max\{V, q\}$ and $5 \le b \le 10$.*

For the proof, we argue closely along the lines of Huckleberry-Sauer [28] and Szabo [42] (see [4]). But we use the centralizer Lemma 5.5 instead. Let $H \trianglelefteq G$ be a maximum among normal subgroups of G which acts freely on X. Take a normal subgroup $K/H \trianglelefteq G/H$. Fix some $\tau \in K$ such that the fixed locus $X^\tau \neq \emptyset$. We may assume that $\mathrm{ord}(\tau) = \mathrm{ord}(\overline{\tau}) = p$ is a prime (see 5.3 (iii) and Lemma 2.5).

Claim 5.7. *Let $G_1 = C_G(\tau)$. Then $|G| \le |G_1||K|$.*

Proof. Note that $\varphi : G \to K$ ($g \mapsto g^{-1}\tau g$) is a well-defined map. Then the claim follows from that $|G/G_1| = |\varphi(G)| \le |K|$. □

Suppose that we can choose K/H to be abelian. Note that K/H acts on the smooth minimal n-fold X/H of general type. By Theorem 2.11, we have $|K/H| \le x_n K_{X/H}^n = x_n V/|H|$. Thus by Claim 5.7 and Lemmas 5.5 and 2.3, we have $|G| \le |G_1||K/H||H| \le k_n q^2 V^{3+\varepsilon} x_n V \le d_n q^2 V^5$ with $d_n = k_n x_n$.

We may assume that there is no such abelian K/H. Then G/H is semi-simple, i.e., it has no non-trivial abelian normal subgroup. Let M/H be a non-trivial minimal normal subgroup. Then $M/H = \prod^k E/H$, a k-fold product of the same non-abelian simple group E/H. Let M_i/H be all distinct non-trivial minimal normal subgroups of G/H and let $S/H = \prod M_i/H$ be the sockel of G/H. We write $M_i/H = \prod^{k_i} E_i/H$ with non-abelian simple group E_i/H.

Suppose that $M/H = \prod^k E/H$ with $k \ge 2$. Then $\alpha(M) \mid |H|$ in notation of Lemma 5.4, because for every prime factor p_1 of $|M|$, either $p_1 \mid |H|$, or $p_1 \mid |M/H|$ and hence $p_1^k \mid |M/H|$ (and $p_1^2 \mid |G|$). So by Lemma 5.4, $\exp(M)$ divides $4q^2\alpha(M)$ and also $4q^2|H|$. Our Lemma 2.3 implies that $|M| \le \exp(M) K_X^n$. Substituting all these in and applying Lemmas 5.5 and 2.3, we obtain $|G| \le |G_1||M| \le k_n q^2 V^{3+\varepsilon} 4q^2 |H| V \le 4k_n q^4 V^6$ and we are done.

So we may assume that $M/H = E/H$ is non-abelian simple for every non-trivial minimal normal $M/H \trianglelefteq G/H$. Suppose that M/H is one of the 26 sporadic non-abelian simple groups. Then $M/H \le d$, a constant (the order of the Monster simple group). Thus as above, $|G| \le |G_1||M/H||H| \le k_n q^2 V^{3+\varepsilon} d V \le d k_n q^2 V^5$.

Suppose that E/H is of Lie type. By the proof of Huckleberry-Sauer [28] Proposition 7, there exist a universal constant d and a Sylow p_1-subgroup U/H of M/H such that $|M/H| \le d|U/H|^{5/2}$. Note that U/H acts on the smooth minimal n-fold X/H of general type and the fixed locus $(X/H)^{U/H}$ is finite by 5.3 (iii) and Lemma 2.5. By Lemmas 2.3 and 2.6 and Theorem 2.11, we have $|U/H| \le \exp(U/H) K_{X/H}^n = \mu(U/H) V/|H| \le x_n K_{X/H}^n V/|H| = x_n (V/|H|)^2$. Thus $|M/H| \le d x_n^{5/2} (V/|H|)^5$. By Lemma 5.5 we have $|G| \le |G_1||M| \le k_n q^2 V^{3+\varepsilon} d x_n^{5/2} V^5 = d_n q^2 V^{8+\varepsilon}$ with $d_n = k_n d x_n^{5/2}$. If $p^2 \mid |G|$ then $\varepsilon = 0$ and we are done.

If the fixed locus $X^U = \emptyset$, then $|U| \le V$ by Lemma 2.3 and we will even have a better bound. We may assume that the fixed locus $X^U \ne \emptyset$ (equivalently $(X/H)^{U/H} \ne \emptyset$ by Lemma 2.5). Then our initial $\overline{\tau}$ (of order p) can be taken from U/H so that $p_1 = p$ and U/H is a p-Sylow subgroup of M/H. If p^2 divides $|U/H|$ then it also divides $|G|$, whence $\varepsilon = 0$ and we are done. Otherwise, $|U/H| = p \le x_n K_{X/H}^n = x_n V/|H|$ by Theorem 2.11 and we will be done again.

We are left with the case where each M_i/H is an alternating group A_{k_i}. Suppose that $M/H = A_k$ is the smallest among them. If there are two M_i/H then we will be done as in the case $M/H = \prod^k E/H$ with $k \ge 2$ because the fact that $|M/H|$ divides $|M_i/H|$ for two i implies that $\alpha(M) \mid |H|$ as well.

Therefore, we assume that the sockel $S/H = M/H = A_k$. Then as noticed by Huckleberry and Sauer, the conjugation map induces an injection $1 \to G/H \to \mathrm{Aut}(S/H)$. Hence we have the following, where the first factor 2 is needed only when $k = 6$ (so that $\mathrm{Aut}(A_6)/A_6 = (\mathbb{Z}/(2))^{\oplus 2}$; see Atlas [12]) :

$$|G/H| \le |\mathrm{Aut}(S/H)| \le 2 \times 2\,|A_k| = 4|M/H|.$$

First, treat the case $k \le 58$. Then $|G| \le 4|A_k||H| \le 4|A_k|V \le 4(58!)V$ by Lemma 2.3, and we are done. Or as noticed by Szabo [42], M/H contains a Sylow p_1-subgroup U/H such that (as above) $|M/H| \le |U/H|^5 \le (x_n(V/|H|)^2)^5$, whence $|G| \le d_n V^{10}$ with $d_n = 4x_n^5$.

We next deal with the case $k \ge 59$. We use the approach of Szabo [42], but we apply the centralizer Lemma 2.6 instead. As in Lemma 2.10, set $\ell = [k/4]$ and we have $A_\ell \times A_{3\ell} \le A_k \le A_{4\ell+3}$.

Suppose that the subgroup $A_\ell < A_k = M/H$ acts freely on X/H. Then $\tilde{A}_\ell < G$ (with $\tilde{A}_\ell/H = A_\ell$) also acts freely on X by Lemma 2.5, whence $|\tilde{A}_\ell| \le K_X^n = V$ by Lemma 2.3. By Lemma 2.10, $|A_k| \le |A_\ell|^8 \le (V/|H|)^8$, so $|G| = |G/H||H| \le 4|A_k||H| \le 4V^8$.

Suppose that A_ℓ does not act freely on X/H. We may take $\overline{\tau} \in G/H$ to be from A_ℓ and $\tau \in M$ with $\mathrm{ord}(\tau) = p = \mathrm{ord}(\overline{\tau})$. Now $A_{3\ell} \le C_{M/H}(\overline{\tau}) \le C_{G/H}(\overline{\tau})$. This and Lemma 2.6 imply $|A_{3\ell}| \le |C_G(\tau)| = |G_1|$. By Lemmas 2.10, 2.3 and 5.5, $|G| \le 4|A_k||H| \le 4|A_{3\ell}|^{1.7}V \le 4|G_1|^{1.7}V \le 4(k_n q^2 V^{3+\varepsilon})^{1.7}V = d_n q^{3.4} V^{6.1+1.7\varepsilon}$, where $d_n = 4k_n^{1.7}$. Since p divides $|A_\ell|$, our p^2 divides $A_k = M/H$ and also $|G|$. So $\varepsilon = 0$ and we are done. This completes the proof of Proposition 5.6.

5.8. Proof of Theorem 5.1

Since X is of general type, $G \le \mathrm{Aut}(X)$ is finite. Write $\mathrm{Aut}_{\mathrm{variety}}(A) = T \rtimes A_0$ as at the beginning of the section. Set $T_G = T \cap G$ which acts freely on A and X. If $G = T_G$, then $|G| \le K_X^n$ by Lemma 2.3, and we are done. So assume that $T_G < G$. Note that $A \to A/T_G$ is an isogeny of abelian varieties, $X \to X/T_G$ is etale and $G/T_G \le \{g \in \mathrm{Aut}_{\mathrm{variety}}(A/T_G) \mid g(X/T_G) = X/T_G\}$. We shall check the conditions in 5.3 for $G/T_G \le \mathrm{Aut}(X/T_G)$ and $X/T_G \subset A/T_G$. For every $g \in G \setminus T_G$, we have $A^g \ne \emptyset$ by the assumpton on G, so $(A/T_G)^{\overline{g}} \ne \emptyset$ by Lemma 2.5; here $\overline{g} = gT_G \in G/T_G$. Thus $\overline{g}$ is not a translation on A/T_G. So all conditions in 5.3 are satisfied by G/T_G and $X/T_G \subset A/T_G$; see Lemma 2.5 and the assumption on G. By Proposition 5.6, we have $|G/T_G| \le d_n q(A/T_G)^{10-b}(K_{X/T_G}^n)^b = d_n q^{10-b}(V/|T_G|)^b$. Now the theorem follows because $b \ge 5$. This proves Theorem 5.1.

5.9. Proof of Corollary 5.2

By Lemma 2.9, X is of general type and the fixed locus A^g for every $\mathrm{id} \neq g \in G$ is a non-empty finite set unless g is a translation of A. Thus we can apply Theorem 5.1. Now Corollary 5.2 with $d'_n = d_n e_n^5$, follows from Theorem 5.1, Lemma 2.9, Proposition 4.1 and Theorem 2.8.

6. Proofs of Theorems 1.1 and 1.2

In this section we shall prove Theorems 1.1 and 1.2. We start with an observation which bounds $\deg(\mathrm{alb}_X)$ in terms of the volume of X. Note that $\mathrm{Alb}(X) = \mathrm{Alb}(X')$ if X and X' are smooth and birational. The lemma below follows from Lemma 2.2 and Theorem 2.8 (1) after resolving singularities of W, since the volumes are birational invariants.

Lemma 6.1. *Let X be a smooth projective n-fold. Suppose that* $\mathrm{alb}_X : X \to W := \mathrm{alb}(X) \subset \mathrm{Alb}(X)$ *is generically finite onto W, and W is of general type. Then $V(X) \geq \deg(\mathrm{alb}_X)\ V(W) \geq \deg(\mathrm{alb}_X)$.*

6.2. Proof of Theorem 1.2

Let X and $G = \mathrm{Bir}(X)$ be as in Theorem 1.2. By Hanamura [23] Lemma 2.4, after a smooth modification of X, we may assume that G acts regularly on X. The universal property of $\mathrm{Alb}(X)$ implies that there is an induced action (not necessarily faithful) of G on $\mathrm{Alb}(X)$ such that $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is G-equivariant. Let K (resp. $\overline{G}$) be the kernel (resp. image) of the homomorphism $G = \mathrm{Aut}(X) \to \mathrm{Aut}_{\mathrm{variety}}(\mathrm{Alb}(X))$ so that we have the exact sequence

$$1 \to K \to G \to \overline{G} \to 1.$$

Since K acts trivially on $W = \mathrm{alb}_X(X)$, we can factor alb_X as $X \to X/K \to W$. In particular, $|K| \leq \deg(\mathrm{alb}_X)$. By Lemma 2.9 and the assumption in Theorem 1.2, the fixed locus $\mathrm{Alb}(X)^{\overline{g}}$ for every $\mathrm{id} \neq \overline{g} \in \overline{G}$ is a non-empty finite set unless $\overline{g}$ is a translation of $\mathrm{Alb}(X)$. Thus, by Theorem 2.8 (2), we can apply Proposition 4.1 and Theorem 5.1 to W. Setting $V = V(X)$ and $V(W) = K_W^n$ and noting that $q := q(W) = q(X) = \dim \mathrm{Alb}(X)$ by the universal property and definition of $\mathrm{Alb}(X)$, one obtains the inequalities below with $d''_n = d_n e_n^{10-b}$:

$$|\overline{G}| \leq d_n q^{10-b} V(W)^b \leq d''_n V(W)^{10}.$$

Now by Lemma 6.1 or 2.2, we conclude Theorem 1.2:

$$|G| = |K||\overline{G}| \leq \deg(\mathrm{alb}_X)|\overline{G}| \leq d''_n V \cdot V(W)^9 \leq d''_n V^{10}.$$

6.3. Proof of Theorem 1.1

Theorem 1.1 (1) is a special case of Theorem 1.1 (2). Therefore, we have only to show Theorem 1.1 (2).

So suppose that alb_X is not generically finite onto a 3-fold of general type. Set $W = \mathrm{alb}_X(X) \subset \mathrm{Alb}(X)$. Then either $\dim W < 3$, or $\kappa(W) < \dim W \le 3$. As above, after a smooth modification of X, we may assume that $G = \mathrm{Bir}(X)$ acts regularly on X and of course on $\mathrm{Alb}(X)$ (not necessarily faithful on the latter) so that $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is G-equivariant. Since $\dim \mathrm{Alb}(X) = q(X) \ge 4 > \dim X \ge \dim W$ by the assumption, our W is a proper subvariety of $\mathrm{Alb}(X)$. Hence by Ueno [43] Lemma 9.14 and Corollary 10.4, the Kodaira dimension $\kappa(W) \ge 1$. As in Ueno [43] Theorem 10.9 or Mori [39] Theorem 3.7, let B be the identity connected component of $\{a \in \mathrm{Alb}(X) \,|\, a + W \subseteq W\}$. Then W/B is of general type, $W \to W/B$ is an etale fibre bundle with fibre B and is birational to the Iitaka fibring of W. Since the pluri-canonical systems of W are G-stable, we may take G-equivariant desingularizations $X' \to X$ and $Y' \to W/B$ such that $X' \to Y'$ is a well-defined G-equivariant morphism, though G might not act faithfully on Y'. Rewrite X' as X. Take a Stein factorization $X \to Y \to Y'$ so that $f : X \to Y$ has connected general fibre F and is necessarily G-equivariant. We may also assume that Y is already smooth (or do equivariant modifications again). Note that $G = \mathrm{Bir}(X) = \mathrm{Aut}(X)$ now. We have $\dim Y = \dim Y' = \kappa(W) = 1, 2$. Since the subvariety $W/B < \mathrm{Alb}(X)/B$ is of general type, both Y and Y' are of general type. Apply Theorem 3.1 to the G-equivariant map $f : X \to Y$ with connected general fibre F say. Note that $3 = \dim X = \dim Y + \dim F$ and $|\mathrm{Bir}(Z)| = |\mathrm{Aut}(Z_{\min})| \le (42)^{\dim Z} V(Z)$ for both $Z = Y$ and $Z = F$, thanks to Hurwitz and Xiao. Here $Z_{\min}$ is the *unique* smooth minimal model of Z, since $\dim Z \le 2$. By Theorem 3.1, we have $|G| \le a\ V(X)$ with $a = (42)^3/3$. This proves Theorem 1.1 (2).

References

[1] D. Abramovich, Subvarieties of semiabelian varieties. Compositio Math. 90 (1994), no. 1, 37–52.

[2] U. Angehrn and Y. Siu, Effective freeness and point separation for adjoint bundles. Invent. Math. 122 (1995), no. 2, 291–308.

[3] E. Aljadeff and J. Sonn, Bounds on orders of finite subgroups of $\mathrm{PGL}_n(K)$. J. Algebra 210 (1998), no. 1, 352–360.

[4] M. Aschbacher, The status of the classification of the finite simple groups. Notices Amer. Math. Soc. 51 (2004), no. 7, 736–740.

[5] W. Barth, C. Peters and A. Van de Ven, Compact complex surfaces. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 4. Springer-Verlag, Berlin, 1984.

[6] C. Birkenhake and H. Lange, Complex abelian varieties. Second edition. Grundlehren der Mathematischen Wissenschaften, 302. Springer-Verlag, Berlin, 2004.

[7] E. Bombieri, Canonical models of surfaces of general type. Inst. Hautes tudes Sci. Publ. Math. No. 42, (1973), 171–219.

[8] J. Cai, On abelian automorphism groups of 3-folds of general type. Algebra Colloq. 2 (1995), no. 4, 373–382.

[9] F. Catanese and M. Schneider, Polynomial bounds for abelian groups of automorphisms. Special issue in honour of Frans Oort. Compositio Math. 97 (1995), no. 1-2, 1–15.

[10] Jungkai A. Chen and C. D. Hacon, Linear series of irregular varieties. Algebraic geometry in East Asia (Kyoto, 2001), 143–153, World Sci. Publishing, River Edge, NJ, 2002.

[11] M. Chen, Inequalities of Noether type for 3-folds of general type. J. Math. Soc. Japan 56 (2004), no. 4, 1131–1155.

[12] J. H. Conway, R. T. Curtis, S. P. Norton, R . A. Parker and R. A. Wilson, Atlas of finite groups. Oxford University Press, Eynsham, 1985.

[13] A. Corti, Polynomial bounds for the number of automorphisms of a surface of general type. Ann. Sci. cole Norm. Sup. (4) 24 (1991), no. 1, 113–137.

[14] J. -P. Demailly, T. Peternell and M. Schneider, Compact complex manifolds with numerically effective tangent bundles. J. Algebraic Geom. 3 (1994), no. 2, 295–345.

[15] S. Friedland, The maximal orders of finite subgroups in $\mathrm{GL}_n(Q)$. Proc. Amer. Math. Soc. 125 (1997), no. 12, 3519–3526.

[16] T. Fujita, On Kahler fiber spaces over curves. J. Math. Soc. Japan 30 (1978), no. 4, 779–794.

[17] W. Fulton, Intersection theory. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 2. Springer-Verlag, Berlin, 1984.

[18] D. Gorenstein, Finite groups. Harper & Row, Publishers, New York-London 1968.

[19] P. Griffiths and J. Harris, Principles of algebraic geometry. John Wiley & Sons, Inc., New York, 1994.

[20] P. Griffiths and J. Harris, Algebraic geometry and local differential geometry. Ann. Sci. cole Norm. Sup. (4) 12 (1979), no. 3, 355–452.

[21] C. D. Hacon, On the degree of the canonical maps of 3-folds. Proc. Japan Acad. Ser. A Math. Sci. 80 (2004), no. 8, 166–167.

[22] C. D. Hacon and J. McKernan, Boundedness of pluricanonical maps of varieties of general type, math.AG/0504327.

[23] M. Hanamura, The birational automorphism groups and the Albanese maps of varieties with Kodaira dimension zero. J. Reine Angew. Math. 411 (1990), 124–136.

[24] R. Hartshorne, Ample vector bundles on curves. Nagoya Math. J. 43 (1971), 73–89.

[25] R. Hartshorne, Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[26] G. Heier, Effective finiteness theorems for maps between canonically polarized compact complex manifolds. Math. Nachr. 278 (2005), no. 1-2, 133–140.

[27] A. Howard and A. J. Sommese, On the orders of the automorphism groups of certain projective manifolds. Manifolds and Lie groups (Notre Dame, Ind., 1980), pp. 145–158, Progr. Math., 14, Birkhauser, Boston, Mass., 1981.

[28] A. T. Huckleberry and M. Sauer, On the order of the automorphism group of a surface of general type. Math. Z. 205 (1990), no. 2, 321–329.

[29] Y. Kawamata, A generalization of Kodaira-Ramanujam's vanishing theorem. Math. Ann. 261 (1982), no. 1, 43–46.

[30] Y. Kawamata, Kodaira dimension of algebraic fiber spaces over curves. Invent. Math. 66 (1982), no. 1, 57–71.

[31] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem. Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[32] J. Kollar, Higher direct images of dualizing sheaves. I. Ann. of Math. (2) 123 (1986), no. 1, 11–42.

[33] J. Kollar, Singularities of pairs. Algebraic geometry—Santa Cruz 1995, 221–287, Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, 1997.

[34] J. Kollar and S. Mori, Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti. Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998.

[35] S. J. Kovacs, Number of automorphisms of principally polarized abelian varieties. Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), 3–7, Contemp. Math., 276, Amer. Math. Soc., Providence, RI, 2001.

[36] R. Lazarsfeld, Positivity in algebraic geometry. I; II. Classical setting: line bundles and linear series. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 48; 49. Springer-Verlag, Berlin, 2004.

[37] S. Lee, Quartic-canonical systems on canonical threefolds of index 1. Special issue in honor of Robin Hartshorne. Comm. Algebra 28 (2000), no. 12, 5517–5530.

[38] Y. Miyaoka, The Chern classes and Kodaira dimension of a minimal variety. Algebraic geometry, Sendai, 1985, 449–476, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[39] S. Mori, Classification of higher-dimensional varieties. Algebraic geometry, Bowdoin, 1985, 269–331, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[40] Z. Ran, The structure of Gauss-like maps. Compositio Math. 52 (1984), no. 2, 171–177.

[41] M. Reid, Young person's guide to canonical singularities. Algebraic geometry, Bowdoin, 1985, 345–414, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[42] E. Szabo, Bounding automorphism groups. Math. Ann. 304 (1996), no. 4, 801–811.

[43] K. Ueno, Classification theory of algebraic varieties and compact complex spaces. Notes written in collaboration with P. Cherenack. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975.

[44] K. Ueno, A remark on automorphisms of Enriques surfaces. J. Fac. Sci. Univ. Tokyo Sect. I A Math. 23 (1976), no. 1, 149–165.

[45] E. Viehweg, Vanishing theorems. J. Reine Angew. Math. 335 (1982), 1–8.

[46] E. Viehweg, Die Additivitat der Kodaira Dimension fur projektive Faserraume uber Varietaten des allgemeinen Typs. J. Reine Angew. Math. 330 (1982), 132–142.

[47] B. Weisfeiler, Post-classification version of Jordan's theorem on finite linear groups. Proc. Nat. Acad. Sci. U.S.A. 81 (1984), no. 16, Phys. Sci., 5278–5279.

[48] G. Xiao, On abelian automorphism group of a surface of general type. Invent. Math. 102 (1990), no. 3, 619–631.

[49] G. Xiao, Bound of automorphisms of surfaces of general type. I. Ann. of Math. (2) 139 (1994), no. 1, 51–77.

[50] G. Xiao, Bound of automorphisms of surfaces of general type. II. J. Algebraic Geom. 4 (1995), no. 4, 701–793.

[51] G. Xiao, Linear bound for abelian automorphisms of varieties of general type. J. Reine Angew. Math. 476 (1996), 201–207.

[52] S. T. Yau, On the Ricci curvature of a compact Kahler manifold and the complex Monge-Ampre equation. I. Comm. Pure Appl. Math. 31 (1978), no. 3, 339–411.

Department of Mathematics, National University of Singapore
2 Science Drive 2, Singapore 117543, SINGAPORE
E-mail: matzdq@nus.edu.sg

7. Appendix

A PRODUCT FORMULA FOR VOLUMES OF VARIETIES

By Yujiro Kawamata

The volume $v(X)$ of a smooth projective variety X is defined by

$$v(X) = \limsup \frac{\dim H^0(X, mK_X)}{m^d/d!}$$

where $d = \dim X$. This is a birational invariant.

Theorem 7.1. *Let $f : X \to Y$ be a surjective morphism of smooth projective varieties with connected fibers. Assume that both Y and the general fiber F of f are varieties of general type. Then*

$$\frac{v(X)}{d_X!} \geq \frac{v(Y)}{d_Y!}\frac{v(F)}{d_F!}$$

where $d_X = \dim X$, $d_Y = \dim Y$ and $d_F = \dim F$.

Proof. Let H be an ample divisor on Y. There exists a positive integer m_0 such that $m_0K_Y - H$ is effective.

Let ϵ be a positive integer. By Fujita's approximation theorem ([1]), after replacing a birational model of X, there exists a positive integer m_1 and ample divisors L on F such that $m_1K_F - L$ is effective and $v(\frac{1}{m_1}L) > v(F) - \epsilon$.

By Viehweg's weak positivity theorem ([2]), there exists a positive integer k such that $S^k(f_*\mathcal{O}_X(m_1K_{X/Y}) \otimes \mathcal{O}_Y(H))$ is generically generated by global sections for a positive integer k. k is a function on H and m_1.

We have

$$\begin{aligned}
&\text{rank Im}(S^m S^k(f_*\mathcal{O}_X(m_1K_{X/Y})) \to f_*\mathcal{O}_X(km_1mK_{X/Y})) \\
&\geq \dim H^0(F, kmL) \\
&\geq (v(F) - 2\epsilon)\frac{(km_1m)^{d_F}}{d_F!}
\end{aligned}$$

for sufficiently large m.

Then

$$\begin{aligned}
&\dim H^0(X, km_1mK_X)\\
&\geq \dim H^0(Y, k(m_1-m_0)mK_Y) \times (v(F)-2\epsilon)\frac{(km_1m)^{d_F}}{d_F!}\\
&\geq (v(Y)-\epsilon)\frac{(k(m_1-m_0)m)^{d_Y}}{d_Y!}(v(F)-2\epsilon)\frac{(km_1m)^{d_F}}{d_F!}\\
&\geq (v(Y)-2\epsilon)(v(F)-2\epsilon)\frac{(km_1m)^{d_X}}{d_Y!d_F!}
\end{aligned}$$

if we take m_1 large compared with m_0 such that

$$\frac{(v(Y)-\epsilon)}{(v(Y)-2\epsilon)} \geq (\frac{m_1}{m_1-m_0})^{d_Y}.$$

$\square$

Remark 7.2. If $X = Y \times F$, then we have an equality in the formula. We expect that the equality implies the isotriviality of the family.

References

[1] Fujita, Takao. *Approximating Zariski decomposition of big line bundles.* Kodai Math. J. 17 (1994), no. 1, 1–3.

[2] Viehweg, Eckart. *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces.* Algebraic varieties and analytic varieties (Tokyo, 1981), 329–353, Adv. Stud. Pure Math., 1, North-Holland, Amsterdam, 1983.

Department of Mathematical Sciences, University of Tokyo
Komaba, Meguro, Tokyo, 153-8914, JAPAN
E-mail: kawamata@ms.u-tokyo.ac.jp